Optimum Power Allocation in Parallel Poisson Optical Channel

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***Abstract*— In this paper we investigate the Shannon capacity of single and parallel Poisson channels in different scenarios. At first, by examining the capacity equation of the single Poisson channel, the minimum average power that maximizes the capacity is found. Then, applying the method of Lagrange multipliers on the capacity expression of the parallel Poisson channel, we obtain the optimum power allocation maximizing the parallel Poisson channel capacity.**

I. Introduction

The traditional application of optical communications is at backbone networks, but it is attracting growing attention due to its new applications in access networks. In this paper, we accomplish an information theoretic study on optical parallel Poisson channels that have applications in access networks.

In optical intensity-modulated communication systems there are several well-known noises, such as shot noise, thermal noise, and laser intensity noise [1]. When the received optical power is large, the dominant noise is usually the shot noise which is modeled as a Poisson random process [2]. The optical communication channels dominated by shot noise are called Poisson channels that have been studied by many researchers [2-5]. Its conceptual simplicity and the advent of many uncoded and coded communication techniques have propelled an extensive information-theoretic study of communication over this channel to specify the final limits and the ultimate potential of this channel [6]. Using the Shannon capacity expression of Poisson channel derived in [3,4], we obtain the minimum average power maximizing the capacity. Then, we consider a 2-fold parallel Poisson channel and obtain the optimum power allocation scenario that maximizes the capacity of parallel channel. All special cases of this power allocation are thoroughly discussed. Finally, we generalize the power allocation scenario to the *n*-fold parallel Poisson channel. The importance of the parallel channels lies in the study of channels with non-white or colored noise to find the optimum power spectrum for the channel input. The optimum power allocation leads in ‘water-filling’ when applied to the colored Gaussian noise channel [7]. Now, we introduce the notation used in this paper.

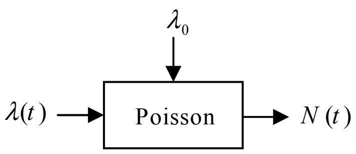

Figure 1. The Poisson channel

The Poisson channel shown in figure (1) is modeled as follows [4]. Let $N(t)$ represent the channel output, which is the number of photoelectrons counted by a direct detection device (photo-detector) in the interval $[0,t]$. It follows that $N(0)=0$.

$N(t)$ has been shown to be a doubly stochastic Poisson process with instantaneous rate $\lambda(t)+\lambda_0$ such that [5]:

$$\Pr\{N(t+\tau)-N(t)=k \mid \lambda(t)\}=\frac{e^{-\Lambda}\Lambda^k}{k!} \quad k=0,1,\ldots$$
$$\Lambda=\int_t^{t+\tau}(\lambda(t')+\lambda_0)dt' \tag{1}$$

where $\lambda(t)\geq 0$, in ‘photons/s’, is the channel input and is called the photon-count intensity and $\lambda_0\geq 0$, also in ‘photons/s’, is a constant representing the dark current and the background noise [1,5]. Note that $\lambda(t)$ and λ_0 are due to the channel input and background noise respectively [6]. $\lambda(t)$ is proportional to the received optical power, $p(t)$, through $\lambda(t)=\eta p(t)/h\nu$ where ν is the optical frequency, η is the quantum efficiency of photo-detector and h is the Plank’s constant [1]. It is assumed that the input signal is peak and average limited so that [3-5]:

$$0\leq\lambda(t)\leq A$$
$$\frac{1}{T}E\left\{\int_t^{t+\tau}\lambda(t')dt'\right\}\leq\sigma A \tag{2}$$

where A and $0\leq\sigma\leq 1$ are constants and E is the statistical expectation operator. The constraints of equation (2) are reasonable because the peak and average optical transmitted power in practical systems, such as fiber optics [1] and free space optical (FSO) systems [6], are limited in order to prevent the fiber core damage and to achieve the eye safety level.

The rest of this paper is organized as follows. In section 2, capacity of the Poisson channel in the general case and in the special cases ‘very noisy’ and ‘noiseless’ is examined. Section 3 is devoted to the parallel Poisson channel. Capacity of the parallel Poisson channel is examined and the optimum power allocation which maximizes this capacity is obtained using the method of Lagrange multipliers. Section 4 concludes the paper.

II. Capacity of the Poisson Channel

In 1978 capacity of the Poisson channel was derived for the special case of $\lambda_0=1$ by Kabanov [3]. Two years later Davis [4] extended Kabanov’s derivation and found the capacity of Poisson channel for an arbitrary $\lambda_0 \geq 0$. Both of them used the Martingale technique to maximize the mutual information between the input $\lambda(t)$ and the output $N(t)$ subject to the probability distribution of $\lambda(t)$. The Shannon capacity of the Poisson channel is given by [4]:

$$C(A,\sigma,s) = A[p(1+s)\ln(1+s) + (1-p)s\ln(s) - (p+s)\ln(p+s)] \quad \text{nats/s} \tag{3}$$

Where $s = \lambda_0/A$ and $p = \min(\sigma, q(s))$ such that:

$$q(s) = \frac{(1+s)^{1+s}}{s^s e} - s \tag{4}$$

An alternative derivation of the above equation which was more pleasant to information theorists was given by Wyner [5] in 1988. His derivation is based on the random coding technique due to Shannon [7].

Now we examine equation (3) in two extreme cases 'very noisy' and 'noiseless'. In the noiseless channel, dark current and background noise constant, λ_0, is equal to zero, so that $s = 0$. Substituting $s = 0$ in equation (3), we will obtain the following equation for the capacity of noiseless Poisson channel [5]:

$$C = -A\,p\ln(p) \qquad \text{nats/s} \tag{5}$$

where $p = \min(\sigma, e^{-1})$.

When the channel is very noisy, λ_0 is very large compared to A so $s \to \infty$ and the corresponding capacity will be [4]:

$$C = A\frac{p(1-p)}{2s} + o(s^{-1}) \qquad \text{nats/s} \tag{6}$$

Where $p = \min(\sigma, 0.5)$ and 'o' is a function having the property $\lim_{s\to\infty} s\,o(s^{-1}) = 0$.

It can be easily shown that capacity of the Poisson channel given in equation (3), is an increasing function of p. Since $p = \min(\sigma, q(s))$, it is obvious that $p \leq q(s)$ therefore, considering equation (4) we have $(1+s)^{1+s}/s^s e(p+s) \geq 1$. On the other hand from equation (3) we have:

$$\frac{dC}{dp} = A\ln\left(\frac{(1+s)^{1+s}}{s^s e(p+s)}\right) \tag{7}$$

So $dC/dp \geq 0$ which proves that the capacity is increasing in p. Therefore, the maximum capacity of Poisson channel is achieved when the parameter p has its maximum value. Accordingly, the maximum capacity of noiseless Poisson channel is equal to Ae^{-1} nats/s which is obtained from equation (5) by letting $p = e^{-1}$. One may intuitively expect that increasing the channel input average power must lead to a larger capacity. In other words for the Poisson channel with peak intensity rate A and background noise λ_0, it seems that the larger σ is, the larger the capacity will be. But according to (4), as far as $\sigma \geq q(s)$, the capacity is independent of σ and is equal to:

$$C(A,\sigma = q(s),s) = A\left[\frac{(1+s)^{1+s}}{s^s e} - s(1+s)\ln\left(1+\frac{1}{s}\right)\right] \quad \text{nats/s} \tag{8}$$

Therefore, the minimum average power which maximizes the capacity of Poisson channel is achieved when $\sigma = q(s)$.

Figure (2) shows the capacity of Poisson channel versus the signal-to-noise ratio ($SNR = s^{-1} = A/\lambda_0$) for several values of σ and $A = 10^{12}$ photons/s which is a typical value for the count intensity peak A. Note that the capacity has been expressed in bits/s and has been obtained by dividing equation (3) by $\ln(2)$.

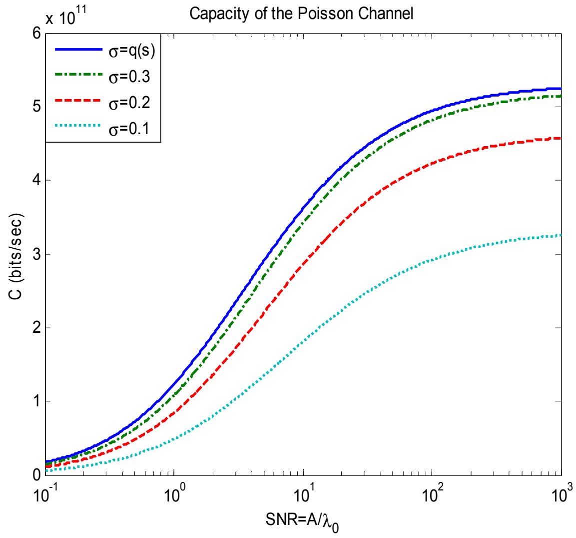

Figure 2. Capacity of the Poisson channel when $A = 10^{12}$ photons/s for several values of σ

Figure (2) also shows the asymptotic behavior of the Poisson channel capacity for large SNR values. As it can be realized from figure (2), unlike the Gaussian channel the capacity of the Poisson channel cannot be increased unboundedly by strengthening the channel input power.

In figure (3) the minimum required count intensity, $q(s)A$, that maximizes the Poisson channel capacity is sketched versus the signal-to-noise ratio when $A = 10^{12}$ photons/s.

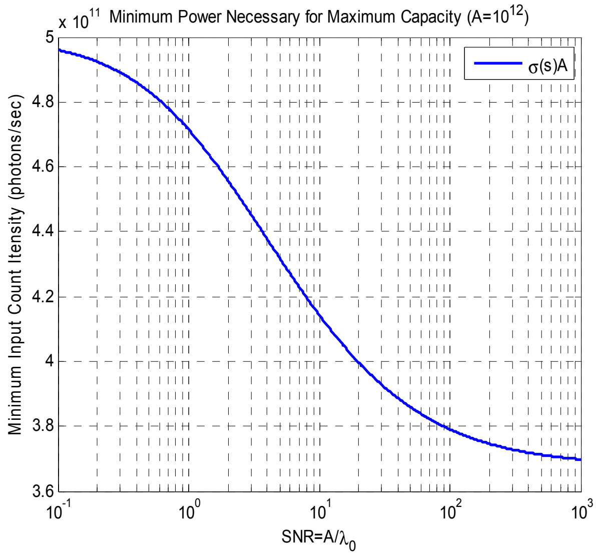

Figure 3. Minimum input count intensity for maximum capacity when $A = 10^{12}$ photons/s.

From figure (3) it is understood that as the background noise decreases the minimum power required to achieve the maximum capacity of the Poisson channel reduces.

III. PARALLEL POISSON CHANNEL

An n-fold parallel channel consists of n independent single channels. The independence of channels is due to their independent noise sources. So, given the inputs of the channels, their outputs are statistically independent.

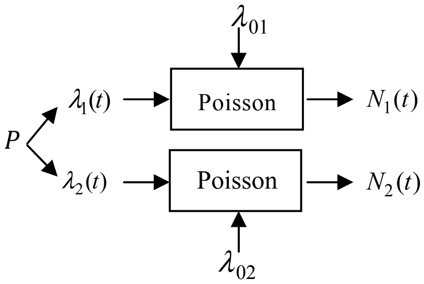

Figure 4. The parallel Poisson channel

For example, for the 2-fold parallel Poisson channel shown in figure (4), we have:

$$\Pr\{N_1, N_2 \mid \lambda_1, \lambda_2\} = \Pr\{N_1 \mid \lambda_1\} \Pr\{N_2 \mid \lambda_2\} \tag{9}$$

Capacity of the aforementioned channel is equal to $C_1 + C_2$ where C_1 and C_2 are the capacities of the individual single channels [7].

Consider a 2-fold parallel Poisson channel with the individual single channel parameters (A_1, λ_{01}) and (A_2, λ_{02}). Assume that a user with average count intensity P (that is proportional to average transmit power) sends its data through the parallel channel. The question is how can this power be allocated to each single channel so that the whole capacity of the parallel channel is maximized?

It is clear that the maximum of P is $A_1 + A_2$ and we must maximize the capacity of parallel channel $C(\sigma_1, \sigma_2)$ subject to the constraint $P = \sigma_1 A_1 + \sigma_2 A_2$. In other words, we have the following optimization problem:

$$\begin{aligned} &Maximize \quad C = C(\sigma_1, \sigma_2) \\ &Subject\ to \quad G(\sigma_1, \sigma_2) = \sigma_1 A_1 + \sigma_2 A_2 = P \end{aligned} \tag{10}$$

According to the maximization method of Lagrange multipliers [7], $\nabla C = \gamma \nabla G$, we have the following set of two simultaneous equations:

$$\frac{\partial C}{\partial \sigma_1} = \gamma A_1 \quad , \quad \frac{\partial C}{\partial \sigma_2} = \gamma A_2 \tag{11}$$

Solving the above equations, we obtain the optimum average to peak ratios of input powers as follow:

$$\begin{aligned} \sigma_1 &= (e^{-\gamma} q(s_1) - (1 - e^{-\gamma}) s_1)^+ \\ \sigma_2 &= (e^{-\gamma} q(s_2) - (1 - e^{-\gamma}) s_2)^+ \end{aligned} \tag{12}$$

Where

$$e^{-\gamma} = \frac{s_1 A_1 + s_2 A_2 + P}{(s_1 + q(s_1)) A_1 + (s_2 + q(s_2)) A_2} \tag{13}$$

Here $(x)^+$ denotes the positive part of x, i.e. it is equal to x for $x > 0$ and is 0 when $x \le 0$.

Equation (12) is a general solution for the problem. Now we consider the special cases in which σ_1 and σ_2 may become zero. Substituting equation (13) into equation (12) under condition $e^{-\gamma} q(s_1) - (1 - e^{-\gamma}) s_1 \le 0$ we obtain the following expression:

$$\frac{s_2 + P/A_2}{s_2 + q(s_2)} \le \frac{s_1}{s_1 + q(s_1)} \Rightarrow \sigma_1 = 0 \,, \sigma_2 = \frac{P}{A_2} \tag{14}$$

In the same way for $e^{-\gamma} q(s_2) - (1 - e^{-\gamma}) s_2 \le 0$ we have:

$$\frac{s_1 + P/A_1}{s_1 + q(s_1)} \le \frac{s_2}{s_2 + q(s_2)} \Rightarrow \sigma_1 = \frac{P}{A_1} \,, \sigma_2 = 0 \tag{15}$$

From equation (13) it is obvious that if $P = q(s_1) A_1 + q(s_2) A_2$, then $e^{-\gamma} = 1$ and $\gamma = 0$. Therefore, considering equation (12), $\sigma_1 = q(s_1)$ and $\sigma_2 = q(s_2)$ are the optimum average to peak ratios.

Although for the case of $P \ge q(s_1) A_1 + q(s_2) A_2$ any allocation in which $\sigma_1 \ge q(s_1)$ and $\sigma_2 \ge q(s_2)$ can be an optimal solution for the problem, we propose the following solution which is in line with the case $\gamma = 0$ in which $\sigma_1 / \sigma_2 = q(s_1) / q(s_2)$:

$$\sigma_1 = \frac{q(s_1) P}{q(s_1) A_1 + q(s_2) A_2} \,, \sigma_2 = \frac{q(s_2) P}{q(s_1) A_1 + q(s_2) A_2} \tag{16}$$

Now, we summarize the results of the problem in the following table.

TABLE 1.
Optimum Power Allocation Results

Condition on Power and Channel	Optimum values of σ_1 and σ_2
$q(s_1)A_1 + q(s_2)A_2 \le P \le A_1 + A_2$	$\sigma_1 = \frac{q(s_1)P}{q(s_1)A_1 + q(s_2)A_2}$ $\sigma_2 = \frac{q(s_2)P}{q(s_1)A_1 + q(s_2)A_2}$
$\frac{s_2 + P/A_2}{s_2 + q(s_2)} \le \frac{s_1}{s_1 + q(s_1)}$	$\sigma_1 = 0 \,, \sigma_2 = \frac{P}{A_2}$
$\frac{s_1 + P/A_1}{s_1 + q(s_1)} \le \frac{s_2}{s_2 + q(s_2)}$	$\sigma_1 = \frac{P}{A_1} \,, \sigma_2 = 0$
Otherwise	$\sigma_1 = e^{-\gamma} q(s_1) - (1 - e^{-\gamma}) s_1$ $\sigma_2 = e^{-\gamma} q(s_2) - (1 - e^{-\gamma}) s_2$ $e^{-\gamma} = \frac{s_1 A_1 + s_2 A_2 + P}{(s_1 + q(s_1)) A_1 + (s_2 + q(s_2)) A_2}$

Figure (5) shows the optimum power allocation in a parallel Poisson channel with parameters $A_1 = 10^9, A_2 = 10^{12}$ photons/s, $s_1 = 0.1$ and $s_2 = 0.3$. From figure (5), it can be observed that for small power levels, the first channel which has a smaller noise ratio is dominant and the whole power is dedicated to it but as the input average power increases, more and more portion of it is given to the second channel which has the greater peak power.

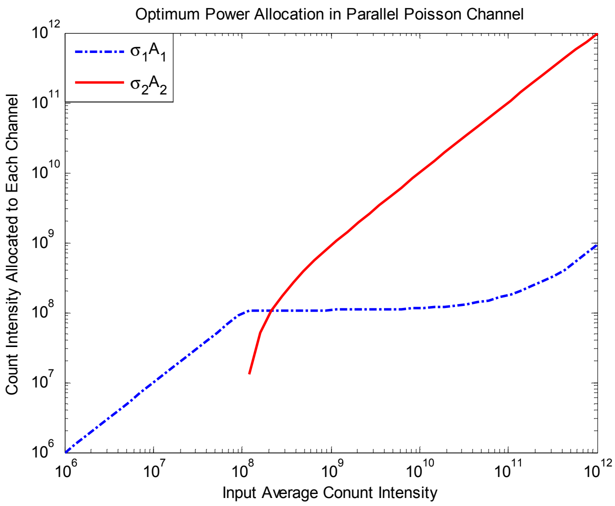

Figure 5. Optimum allocation of count intensity for parallel Poisson channel with parameters $A_1 = 10^9, A_2 = 10^{12}$ photons/s, $s_1 = 0.1$ and $s_2 = 0.3$

Finally, it should be mentioned that the generalization of this method to the n-fold parallel Poisson channel is easy and straightforward. It is due to the generality of the method of Lagrange's multipliers. Assume that there is an n-fold parallel Poisson channel in which the ith channel is characterized by (A_i, σ_i, s_i). Following the same procedure as that of 2-fold case the optimum power allocation for an n-fold parallel Poisson channel is obtained as follows;

$$\sigma_i = (e^{-\gamma} q(s_i) - (1 - e^{-\gamma}) s_i)^+ \quad \mathrm{i} = 1,2, \ldots, n$$
$$e^{-\gamma} = \frac{P + \sum_{i=1}^{n} s_i A_i}{\sum_{i=1}^{n} (s_i + q(s_i)) A_i} \tag{17}$$

Equation (17) is the general form of the optimum power allocation for n-fold parallel Poisson channel and like 2-fold channel we can derive its special cases in which some σ_i's may be zero.

IV. CONCLUSION

The Shannon capacities of single and parallel Poisson channels were examined and the minimum average power that maximizes the capacity of single Poisson channel was calculated. Then using the Lagrange multipliers method, the optimum power allocation maximizing the capacity of parallel Poisson channel was obtained for a 2-fold parallel Poisson channel. Different cases of this allocation were discussed and summarized in a table. It was shown that there are two major different cases, namely, low power and high power. If the average power is low, more power must be dedicated to the channel with low noise, and if the average power is large, more power must be dedicated to the channel with larger peak power. The optimum power allocation scenario for n-fold parallel Poisson channel was also obtained.

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